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Differential formulation of the dynamic renormalisation group in the large-n limit

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Abstract. A differential formulation of the dynamic renormalisation group in the large-n limit is presented for the time-dependent generalisation of the n-vector classical model with purely relaxational dynamics, for both conserved and non-conserved order parameter. A discussion of the dynamical properties with the same steps used in the traditional perturbative approaches is made in terms of which known and new explicit results are obtained. Furthermore, a mechanical analogy is introduced which may give new information about the structure of the dynamic renormalisation group in the large-n limit. This relies on the possibility of using the well known geometrical and mathematical techniques from the Hamilton–Jacobi theory for classical mechanics.

1. Introduction

It is well established (Nicoll *et al* 1975, 1976, Nicoll and Chang 1978, Nelson 1975, Rudnik and Nelson 1976) that differential formulations of the renormalisation group (RG) are far more convenient and efficient to apply than the finite recursion relation approaches. This is especially the case when one manipulates over large domains of the variables involved and when a nonlinear study of the RG properties is made (Nicoll *et al* 1975). With this in mind, we have recently given a differential formulation of the static RG (SRG) in the large-n limit for the n-vector classical model (Busiello *et al* 1981, referred to hereafter as I) and for a wide class of quantum systems (Busiello *et al* 1983, referred to hereafter as II). In any case, it comes out that the SRG in the large-n limit is specified by a first-order 'quasi-linear' partial differential equation. Due to the possibility of using the many powerful techniques familiar from the general theory of partial differential equations (Courant and Hilbert 1962, Garabedian 1964), several advantages with respect to the original approach (Ma 1973, 1974; see also II) appear:

(i) the RG equation can be solved for general rescaling parameter b;

(ii) it becomes simpler to investigate the possibility of multiple solutions, a question already raised by Ma (1974);

(iii) it becomes possible to test directly the validity of the expansions in inverse powers of b used in the original finite recursion relation formulation;

(iv) the standard steps involved in the differential perturbative approach (Ma 1976, Pfeuty and Toulouse 1977, Patashinskii and Prokovski 1979) can be used for studying the critical behaviour in a more natural way;

(v) new explicit results can be obtained.

Thus, from our investigation (see I and II) it clearly emerges that the proposed differential formulation can be conveniently utilised in clarifying some still open questions (Ma 1974) on the structure of the SRG in the large-n limit. On these grounds, it becomes also of methodological interest to extend to dynamics the scheme already developed for statics.

In this paper we present a differential formulation of the dynamic RG (DRG) in the large-n limit for the time-dependent generalisation of the n-vector classical model with purely relaxational dynamics, both for conserved and non-conserved order parameter. This is realised starting from finite recursion relations recently obtained by Szépfalusy and Tél (1980a, b) with the use of a Wilson-type RG transformation. Furthermore, some peculiar aspects of the RG differential equations are pointed out which allow us to develop an interesting mechanical analogy and to introduce an alternative geometrical picture for both the DRG and SRG in the large-n limit. Such an eventuality may be, in our opinion, particularly relevant for obtaining a more intuitive insight about the structure of the DRG, the general features of which are at present less understood than those of the static one.

The paper is organised as follows. In § 2, after an outline of the essential aspects of the Szépfalusy and Tél (ST) treatment which are relevant for us, we introduce the mentioned DRG differential formulation. A discussion of the differential RG equations is made in § 3. Section 4 is devoted to presenting the analogy between Hamiltonian classical mechanics and the DRG in the large-n limit. Finally, in § 5, some conclusions are drawn.

2. Differential DRG equations for local coupling parameters

We consider the time generalisation of an n-vector d-dimensional classical model whose effective Hamiltonian is

$$\mathscr{H}\{\phi\} = \int \mathrm{d}^d x[(\nabla\phi)^2 + U(\phi^2)] \tag{1}$$

where $\phi \equiv \{\phi_\alpha; \alpha = 1, \ldots, n\}$ denotes the n-component order parameter with wavevector cut-off assumed equal to unity and $U(\phi^2)$ is a power series of ϕ^2, conveniently expressed as

$$U(\phi^2) = \sum_{m=1}^{\infty} \frac{u_{2m,2}}{m}(\phi^2)^m, \qquad u_{2m,2} = \mathrm{O}(n^{1-m}). \tag{2}$$

The evolution of the time-dependent order parameter $\phi(\boldsymbol{x}, \tau)$ in the purely relaxational case is governed by the Langevin-type equation (Ma 1976, Hohenberg and Halperin 1977)

$$\partial\phi_\alpha/\partial\tau = -L\,\delta\mathscr{H}/\delta\phi_\alpha + \zeta_\alpha \qquad (\alpha = 1, \ldots, n) \tag{3}$$

where $\zeta(\boldsymbol{x}, \tau) \equiv \{\zeta_\alpha(\boldsymbol{x}, \tau); \alpha = 1, \ldots, n\}$ is a Gaussian white noise and $L = \Gamma_0(\mathrm{i}\nabla)^c$ ($c = 0, 2$). Here Γ_0 is a real constant, conveniently assumed equal to unity, and $c = 0$ ($c = 2$) corresponds to a 'non-conserved' ('conserved') order parameter. The stochastic process (3) can be described equivalently (Machlup and Onsager 1953, Onsager and Machlup 1953, Graham 1973, Janssen 1976, Bausch *et al* 1976, De Dominicis and Peliti 1978) in terms of the path probability functional $P\{\phi(\boldsymbol{x}, \tau)\} \propto \int \mathscr{D}[\tilde{\phi}] \exp \mathscr{A}\{\tilde{\phi}, \phi\}$, where $\mathscr{A}\{\tilde{\phi}, \phi\}$ is an appropriate 'action' and $\tilde{\phi}(\boldsymbol{x}, \tau)$ is an n-vector 'response field' conjugate to ϕ (Martin *et al* 1973).

In the ST formulation of the DRG in the large-n limit, the initial 'large-n action' is of the form (Szépfalusy and Tél 1980b)

$$\mathscr{A}\{\tilde{\phi}, \phi\} = \int \mathrm{d}^d x \int \mathrm{d}\tau \left\{ \sum_{\alpha=1}^{n} \left[-\tilde{\phi}_\alpha L \tilde{\phi}_\alpha + \mathrm{i}\tilde{\phi}_\alpha \left(\frac{\partial \phi_\alpha}{\partial \tau} - L\nabla^2 \phi_\alpha \right) \right] + \varphi t(\phi^2) \right\} \tag{4}$$

where

$$\varphi(\boldsymbol{x}, \tau) = \mathrm{i} \sum_{\alpha=1}^{n} \tilde{\phi}_\alpha L \phi_\alpha + \frac{n}{2} \frac{K_d}{d+c}, \qquad t(\phi^2) = \frac{\mathrm{d}U}{\mathrm{d}\phi^2}, \qquad K_d = \frac{2^{-(d-1)} \pi^{-d/2}}{\Gamma(d/2)}.$$

After the Wilson RG transformation, defined by integrating the weight functional $\exp \mathscr{A}\{\tilde{\phi}, \phi\}$ over field variables with wavevectors in the shell $(b^{-1}, 1)$ and by rescaling of the remaining fields, an infinite number of new coupling parameters arises in the renormalised action. It has been shown (Szépfalusy and Tél 1979, 1980a) that the full parameter space for the dynamics is spanned by two distinct sets of coupling parameters generated by the RG transformation.

(a) 'Local' couplings (not dependent on the variables $\boldsymbol{x}$ and τ) contained only in a part of the general action. By using the simplifying features of the large-n limit as in the Ma approach for statics, it is found, not perturbatively, that they transform among themselves. Thus, in the large-n limit, the local coupling parameters form an invariant subspace (the local parameter space) of the full dynamic parameter space.

(b) 'Non-local' couplings whose dependence on $\boldsymbol{x}$ and τ, generated by the RG transformation, can be investigated only perturbatively (Szépfalusy and Tél 1980a). In this paper we refer only to the local parameter space which contains an infinite number of static and dynamic coupling parameters and is rich enough to exhibit also higher-order critical points (Szépfalusy and Tél 1980a, b). The key point of the ST investigation is that the local parameter space for the model under study in the disordered phase is spanned for large n by an action (part of a more general action) of the form

$$\mathscr{A}\{\tilde{\phi}, \phi\} = \int \mathrm{d}^d x \int \mathrm{d}\tau \left\{ \sum_{\alpha=1}^{n} \left[-\tilde{\phi}_\alpha L \tilde{\phi}_\alpha + \mathrm{i} \left(\frac{\partial \phi_\alpha}{\partial \tau} - L\nabla^2 \phi_\alpha \right) \right] + Y(\phi^2, \varphi) \right\}. \tag{5}$$

Initially $Y(\phi^2, \varphi) = \varphi t(\phi^2)$, but in general, after the RG transformation, it is a function of two variables, which cannot be factorised any more, with the restriction

$$Y(\phi^2, 0) \equiv Y(N_c, 0) = \text{constant}, \qquad N_c = \tfrac{1}{2} n K_d/(d-2), \tag{6}$$

as required by 'causality' (Bausch *et al* 1976). The dynamical (local) coupling parameters are defined by the double power series

$$Y(\phi^2, \varphi) = \sum_{m=1}^{\infty} \sum_{1 \leq \nu \leq m} u_{2m,2\nu} \varphi^{\nu} (\phi^2)^{m-\nu} \tag{7}$$

where $u_{2m,2\nu} = \mathrm{O}(n^{1-m})$ for all ν and the sum over ν starts with $\nu = 1$ because, due to (6), $Y(\phi^2, 0)$ is not regarded as a parameter and it is chosen to be zero at the beginning. The coupling parameters for $\nu = 1$ result just the same as in (2) and characterise the statics. All the local coupling parameters are specified by the function

$Y_{0,1}(\phi^2, \varphi) = \partial Y(\phi^2, \varphi)/\partial\varphi$. However, the partial derivative $Y_{1,0}(\phi^2, \varphi) = \partial Y(\phi^2, \varphi)/\partial\phi^2$, with the restriction $Y_{1,0}(\phi^2, 0) \equiv 0$, also enters the problem since the DRG transformation for local parameters inevitably couples $Y_{0,1}$ to $Y_{1,0}$. This is defined, for $d > 2$, by the finite recursion relations (Szépfalusy and Tél 1980b)

$$\begin{aligned} Y'_{0,1}(\phi^2, \varphi) &= b^2 Y_{0,1}[b^{2-d} Q(\phi^2, \varphi) + N_c\, b^{-(d+c)} R(\phi^2, \varphi)], \\ Y'_{1,0}(\phi^2, \varphi) &= b^{4+c} Y_{1,0}[b^{2-d} Q(\phi^2, \varphi) + N_c\, b^{-(d+c)} R(\phi^2, \varphi)], \end{aligned} \tag{8}$$

where

$$\begin{aligned} Q(\phi^2, \varphi) &= \phi^2 - N_c + \frac{n}{2} K_d \int_1^b \mathrm{d}k\, k^{d-1}(k^{c/2} Z^{-1} - k^{-2}), \\ R(\phi^2, \varphi) &= \varphi - \frac{n}{2} K_d \int_1^b \mathrm{d}k\, k^{d-1}\{k^{3c/2}[k^2 + Y'_{0,1}(\phi^2, \varphi)] Z^{-1} - k^c\}, \\ Z &= \{k^c[k^2 + Y'_{0,1}(\phi^2, \varphi)]^2 - 2 Y'_{1,0}(\phi^2, \varphi)\}^{1/2}. \end{aligned} \tag{9}$$

Note that, at $\varphi = 0$, the DRG equations (8), with the identification $Y_{0,1}(\phi^2, 0) \equiv t(\phi^2)$, reduce to the static ones (Ma 1973) and describe therefore the RG transformation of the static coupling parameters $\{u_{2m,2}\}$.

We are now in a position to give the differential version of the finite DRG recursion relations (8). If we define the infinitesimal RG process $\mathscr{R}_{\delta l}$ by writing $b = \mathrm{e}^{\delta l} \simeq 1 + \delta l$, $\delta l \ll 1$, from (9), to leading order in δl, we obtain

$$\begin{aligned} Q_{\delta l}(\phi^2, \varphi) &\simeq \phi^2 - N_c + \frac{n}{2} K_d\{[(1 + Y_{0,1}(\phi^2, \varphi))^2 - 2 Y_{1,0}(\phi^2, \varphi)]^{-1/2} - 1\}\delta l, \\ R_{\delta l}(\phi^2, \varphi) &\simeq \varphi - \frac{n}{2} K_d\{(1 + Y_{0,1}(\phi^2, \varphi))[(1 + Y_{0,1}(\phi^2, \varphi))^2 - 2 Y_{1,0}(\phi^2, \varphi)]^{-1/2} - 1\}\delta l. \end{aligned} \tag{10}$$

By inserting (10) in (8) and by isolating only terms to leading order in δl, it is straightforward to show that the iteration of $\mathscr{R}_{\delta l}$ generates a continuous sequence of quantities

$$\begin{aligned} t_1(l, \psi^2, \theta) &= [Y_{0,1}(l, \phi^2, \varphi)]_{\substack{\phi^2 = N_c\psi^2 \\ \varphi = [(d-2)/(d+c)]N_c\theta}}, \\ t_2(l, \psi^2, \theta) &= [Y_{1,0}(l, \phi^2, \varphi)]_{\substack{\phi^2 = N_c\psi^2 \\ \varphi = [(d-2)/(d+c)]N_c\theta}}, \end{aligned} \tag{11}$$

obeying the system of two 'quasi-linear' first-order partial equations

$$\frac{\partial t_i}{\partial l} + (d-2)(\psi^2 - F)\frac{\partial t_i}{\partial \psi^2} + (d+c)(\theta - G)\frac{\partial t_i}{\partial \theta} = a_i t_i \qquad (i = 1, 2) \tag{12}$$

where l is a parameter describing the progress of renormalisation averaging, $a_1 = 2$, $a_2 = (4 + c)$,

$$F(t_1, t_2) = [(1 + t_1^2) - 2t_2]^{-1/2}, \qquad G(t_1, t_2) = 1 - (1 + t_1)F(t_1, t_2), \tag{13}$$

and we have introduced, for convenience, the new fields $\psi^2 = \phi^2/N_c$, $\theta = [(d+c)/(d-2)]\varphi/N_c$.

The system of equations (12), which has to be solved with the initial conditions

$$t_i(0,\psi^2,\theta)\equiv t_i^{(0)}(\psi^2,\theta)$$
$$=\begin{cases}\displaystyle\sum_{m=1}^{\infty}\sum_{1\le\nu\le m}\nu u^{(0)}_{2m,2\nu}N_c^{m-1}\left(\frac{d-2}{d+c}\right)^{\nu-1}\theta^{\nu-1}(\psi^2)^{m-\nu}, & i=1,\\ \displaystyle\sum_{m=1}^{\infty}\sum_{1\le\nu\le m}(m-\nu)u^{(0)}_{2m,2\nu}N_c^{m-1}\left(\frac{d-2}{d+c}\right)^{\nu}\theta^{\nu}(\psi^2)^{m-\nu-1}, & i=2,\end{cases} \tag{14}$$

constitutes the differential formulation of the DRG for an n-vector relaxational model in the large-n limit. Since, from (7), $t_2(l,\psi^2,0)\equiv 0$, it is immediately seen that the system (12) for $\theta=0$ and with $t_1(l,\psi^2,0)\equiv t(l,\psi^2)=[\partial U(l,\phi^2)/\partial\phi^2]_{\phi^2=N_c\psi^2}$ reduces to the single SRG equation

$$\partial t/\partial l+(d-2)[\psi^2-1/(1+t)]\,\partial t/\partial\psi^2=2t \tag{15}$$

derived and discussed in I. Thus, with the differential formulation (12), many aspects of the dynamics and statics in the large-n limit can be analysed in a unified way. This will be realised in the next sections.

3. Dynamical fixed points, critical surface and stability

The properties of the DRG in the large-n limit can now be discussed with the same steps used in the more traditional RG perturbative approach. The fixed points $\{t_i^*(\psi^2,\theta)\}$ of the differential DRG transformation (12), which are defined by the invariance conditions $\partial t_i^*/\partial l=0$ $(i=1,2)$, are determined as solutions of the system of partial equations

$$(d-2)(\psi^2-F^*)\,\partial t^*/\partial\psi^2+(d+c)(\theta-G^*)\,\partial t_i^*/\partial\theta=a_it_i^* \qquad (i=1,2) \tag{16}$$

where $F^*=F(t_1^*,t_2^*)$ and $G^*=G(t_1^*,t_2^*)$. It is evident that a Gaussian solution $\{t_i^*\equiv 0\}$ (i.e. $Y^*(\phi^2,\varphi)\equiv 0$ in the 'fixed point action') exists for any $d>2$. For the non-trivial fixed points, as already discussed in I and II for the statics, we must distinguish between 'physical fixed points', which preserve the analyticity properties of the original action, and 'mathematical fixed points'. By inspection of (16), it is immediately seen that any non-trivial fixed solution with $(\partial t_i^*/\partial\xi_j)_{\xi_1=1,\xi_2=0}<\infty$ $(i,j=1,2)$ where $\xi_1=\psi^2$ and $\xi_2=\theta$, satisfies the relations

$$t_1^*(1,0)\equiv t^*(1)=0, \qquad t_2^*(1,0)=0. \tag{17}$$

Furthermore, we must have also

$$t_1^*(\psi^2,0)\equiv t^*(\psi^2), \qquad t_2^*(\psi^2,0)\equiv 0, \tag{18}$$

where $t^*(\psi^2)$ is the non-trivial fixed solution of the static equation obtained from (15) with $\partial t^*/\partial l=0$ (see I). Of course, the singular point $(\psi^2=1,\ \theta=0,\ t_i^*=0\ (i=1,2))$ in the space $\{\psi^2,\theta,t_1,t_2\}$ assumes a role similar to that of the point $(\psi^2=1,\ t^*=0)$ in the plane (ψ^2,t^*) of the static case. A study of the system (16), made on the basis of known general techniques (Courant and Hilbert 1962), gives rise to a non-trivial physical fixed solution in a very complicated implicit form, in agreement with the ST results.

However, an advantage of the present differential formulation, with respect to the finite recursion relation approach, is the possibility to obtain for it an 'explicit' representation. This can be realised taking into account the relations (17), (18) and assuming for $Y^*(\phi^2, \varphi)$ the double Taylor series expansion

$$Y(\phi^2, \varphi) = \sum_{m=1}^{\infty} \sum_{1 \leq \nu \leq m} v^*_{2m,2\nu} \varphi^{\nu} (\phi^2 - N_c)^{m-\nu}. \tag{19}$$

By definition, it results that

$$\begin{aligned} t_1^*(\psi^2, \theta) &= \sum_{m=1}^{\infty} \sum_{1 \leq \nu \leq m} a^{(1)}_{2m,2\nu} \theta^{\nu-1} (\psi^2 - 1)^{m-\nu}, \\ t_2^*(\psi^2, \theta) &= \sum_{m=1}^{\infty} \sum_{1 \leq \nu \leq m} a^{(2)}_{2m,2\nu} \theta^{\nu} (\psi^2 - 1)^{m-\nu-1}, \end{aligned} \tag{20}$$

with $a^{(1)}_{2m,2\nu} \equiv a_{2m,2\nu} = \nu v^*_{2m,2\nu} N_c^{m-1} [(d-2)/(d+c)]^{\nu-1}$ and $a^{(2)}_{2m,2\nu} = [(m-\nu)/\nu] \times [(d-2)/(d+c)] a_{2m,2\nu}$. The unknown quantities $a_{2m,2\nu}$ (and therefore $v^*_{2m,2\nu}$ in (19)) can now be determined by integrating for series the system (16). For instance, for $\{a_{2m,2};\ m \geq 1\}$ and $\{a_{2m,4};\ m \geq 2\}$ we obtain the recursion relations

$$\begin{aligned} &a_{2,2} = 0, \qquad a_{4,2} = (4-d)/(d-2), \\ &a_{2m,2} = \frac{d-2}{d-2m} \left[\sum_{k=1}^{m-2} \left(m - (k+1) + \frac{2}{d-2} \right) a_{2(k+1),2} a_{2(m-k),2} \right. \\ &\qquad \left. + \sum_{k=1}^{m-3} [m-(k+1)] a_{2(m-k),2} \right], \qquad m \geq 3, \qquad \left(\sum_{k=1}^{m-3} \ldots \right)_{m=3} = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &\sum_{k=1}^{m} \left(\tilde{a}_{2(m-k+1),2} \sum_{h=1}^{k} (h a_{2(h+2),4} \beta_{k-h+1} + a_{2(h+1),4} \gamma_{k-h+1}) \right. \\ &\qquad \left. + \frac{(d-2)^2}{d+c} k(k+1) a_{2(k+2),2} \beta_{m-k+1} \right) = \frac{(d-2)(4-d)}{d+c} m a_{2(m+1),2}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} &\tilde{a}_{2k,2} = \begin{cases} 1 & \text{for } k = 1, \\ a_{2k,2} & \text{for } k \geq 2, \end{cases} \qquad \beta_k = \begin{cases} 0 & \text{for } k = 1, \\ \tilde{a}_{2(k-1),2} + a_{2k,2} & \text{for } k \geq 2, \end{cases} \\ &\gamma_k = [2(k-1) + (d-6+c)/(d-2)] a_{2k,2} + 2k a_{2(k+1),2}. \end{aligned} \tag{23}$$

Analogous relations can be written for $a_{2m,6}, a_{2m,8}, \ldots$, etc. The successive step for establishing the stability of the physical fixed points and the critical surface can now be realised by a linearisation procedure of the system (12) close to each fixed point.

Very close to a generic fixed point $\{t_i^*\}$, the system of 'quasi-linear' partial equations (12) reduces to the 'linear' one

$$\mathscr{L} \boldsymbol{\tau}(l, \psi^2, \theta) = \boldsymbol{M} \cdot \boldsymbol{\tau}(l, \psi^2, \theta) \tag{24}$$

with the initial condition $\boldsymbol{\tau}(0, \psi^2, \theta) = \boldsymbol{\tau}^{(0)}(\psi^2, \theta)$, where

$$\mathscr{L} \equiv \frac{\partial}{\partial l} + A^* \frac{\partial}{\partial \psi^2} + B^* \frac{\partial}{\partial \theta}, \qquad \boldsymbol{M} = \begin{pmatrix} C_1^* & D_1^* \\ C_2^* & D_2^* \end{pmatrix}, \qquad \boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix},$$
$$\tau_i(l, \psi^2, \theta) = t_i(l, \psi^2, \theta) - t_i^*(\psi^2, \theta), \qquad |\tau_i| \ll 1, \tag{25}$$

and

$$A^* = (d-2)(\psi^2 - F^*), \qquad B^* = (d+c)(\theta - G^*),$$
$$C_i^* = 2\delta_{i,1} - \left\{(d-2)(1+t_1^*)F^{*3}\frac{\partial}{\partial \psi^2} + (d+c)F^*[1-(1+t_1^*)F^{*2}]\frac{\partial}{\partial \theta}\right\}t_i^*, \tag{26}$$
$$D_i^* = (4+c)\delta_{i,2} + F^{*3}\left[(d-2)\frac{\partial}{\partial \psi^2} - (d+c)(1+t_1^*)\frac{\partial}{\partial \theta}\right]t_i^* \qquad (i = 1, 2)$$

(δ_{ij} = Kronecker delta). For the Gaussian fixed point, in (24) one has $\tau_i \equiv t_i$ $(i = 1, 2)$, $A^* = (d-2)(\psi^2 - 1)$, $B^* = (d+c)\theta$, $C_1^* = 2$, $D_1^* = 0$, $C_2^* = 0$, $D_2^* = 4 + c$. Thus, by integration with the method of characteristics, the solution of the system (24), for $l \gg 1$, assumes the form

$$t_1(l, \psi^2, \theta) \simeq \sum_{m=0}^{\infty} \sum_{0 \leq \nu \leq m} \frac{1}{\nu!(m-\nu)!} f^{(1)}_{(m-\nu,\nu)}(1, 0) \exp(\lambda^{(1)}_{m,\nu} l)\theta^\nu (\psi^2 - 1)^{m-\nu},$$
$$t_2(l, \psi^2, \theta) \simeq \sum_{m=1}^{\infty} \sum_{1 \leq \nu \leq m} \frac{1}{\nu!(m-\nu)!} f^{(2)}_{(m-\nu,\nu)}(1, 0) \exp(\lambda^{(2)}_{m,\nu} l)\theta^\nu (\psi^2 - 1)^{m-\nu}, \tag{27}$$

with

$$f^{(i)}_{(m-\nu,\nu)}(1, 0) = [\partial^m t_i^{(0)}(x, y)/\partial x^{m-\nu}\, \partial y^\nu]_{x=1, y=0} \qquad (i = 1, 2)$$

and

$$\lambda^{(1)}_{m,\nu} = \lambda^{(s)}_m - \nu(2+c) \qquad (m \geq 0, 0 \leq \nu \leq m)$$
$$\lambda^{(2)}_{m,\nu} = \lambda^{(1)}_{m,\nu} + (2+c) \qquad (m \geq 1, 1 \leq \nu \leq m) \tag{28}$$

where $\lambda^{(s)}_m = 2(m+1) - md$ $(m \geq 0)$ are the static Gaussian exponents already obtained in I. As we see, since $\lambda^{(1)}_{0,0} = \lambda^{(s)}_0 = 2$, the condition $f^{(1)}_{0,0}(1, 0) = t_1^{(0)}(1, 0) \equiv t^{(0)}(1) = 0$, which is just the well known criticality condition from the statics, specifies the 'critical surface' in the local parameter space. On this we have $\lim_{l \to \infty} t_i(l, \psi^2, \theta) = t_i^*(\psi^2, \theta) \equiv 0$ $(i = 1, 2)$ for dimensionalities $d > 4$ for which it results that

$$\lambda^{(1)}_{m,\nu} < 0 \qquad \text{for } m \geq 1, 0 \leq \nu \leq m,$$
$$\lambda^{(2)}_{m,\nu} < 0 \qquad \text{for } m \geq 1, 1 \leq \nu \leq m. \tag{29}$$

Thus, the trivial fixed point is stable for $d > 4$ and correspondingly we have a Gaussian behaviour with critical exponents $\eta = 0$, $z = 2 + c$, $\nu = 1/\lambda^{(1)}_{0,0} = 1/\lambda^{(s)}_0 = \frac{1}{2}, \ldots,$ etc, where z is the dynamical critical exponent.

In order to investigate the properties associated with the non-trivial physical fixed point, it is sufficient and simple enough to limit ourselves to the expressions

$$t_1^*(\psi^2, \theta) \simeq \frac{4-d}{d-2}(\psi^2 - 1) + \frac{(d-2)(4-d)}{2(d+c)}\theta, \qquad t_2^*(\psi^2, \theta) \simeq \frac{4-d}{d+c}\theta, \tag{30}$$

obtained from (20)–(23) to leading order in (ψ^2-1) and θ, for which the quantities (26) become

$$\begin{aligned}
&A^* \simeq 2(\psi^2-1) - \frac{(d-2)(4-d)^2}{2(d+c)}\theta, \qquad B^* \simeq (4+c)\theta, \\
&C_1^* \simeq (d-2) + \frac{2(4-d)^2}{d-2}(\psi^2-1) - \frac{3(4-d)^2}{d+c}\theta, \qquad C_2^* \simeq \frac{2(4-d)^2}{d+c}\theta, \\
&D_1^* \simeq \frac{(4-d)^2}{2} + \frac{(4-d)^2(d-5)}{(d-2)}(\psi^2-1) + \frac{(4-d)^2[(d-5)^2-3]}{2(d+c)}\theta, \\
&D_2^* \simeq (d+c) + \frac{2(4-d)^2}{d+c}(\psi^2-1) + \frac{(4-d)^2(d-5)}{d+c}\theta.
\end{aligned} \tag{31}$$

In this case, the system (24) can be integrated and, taking into account that $t_2(l, \psi^2, 0) \equiv 0$, we find, for $l \gg 1$,

$$\begin{aligned}
\tau_1(l, \psi^2, \theta) &\simeq e^{(d-2)l} f_1(\psi^2, \theta)\tau_1^{(0)}(1,0) + e^{(d-4)l} f_2(\psi^2, \theta) + e^{(d-6-c)l} f_3(\psi^2, \theta), \\
\tau_2(l, \psi^2, \theta) &\simeq \left[e^{(d-2)l}\frac{(4-d)^2}{d+c}\tau_1^{(0)}(1,0) \right. \\
&\qquad \left. + e^{(d-4)l}\left(\tau_{2(0,1)}^{(0)}(1,0) - \frac{(4-d)^2}{d+c}\tau_1^{(0)}(1,0)\right)\right]\theta,
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
f_1(\psi^2, \theta) &\simeq 1 + \frac{(4-d)^2}{d-2}(\psi^2-1) + \frac{3(4-d)^2[(4-d)^2-4]}{4(d+c)(4+c)}\theta, \\
f_2(\psi^2, \theta) &\simeq \left(\tau_{1(1,0)}^{(0)}(1,0) - \frac{(4-d)^2}{d-2}\tau_1^{(0)}(1,0)\right)(\psi^2-1) \\
&\quad + \frac{(4-d)^2}{2(2+c)}\left(\tau_{2(0,1)}^{(0)}(1,0) + \frac{d-2}{d+c}\tau_{1(1,0)}^{(0)}(1,0) - \frac{2(4-d)^2}{d+c}\tau_1^{(0)}(1,0)\right)\theta, \\
f_3(\psi^2, \theta) &\simeq \left[\tau_{1(0,1)}^{(0)}(1,0) - \frac{(d-2)(4-d)^2}{2(d+c)(2+c)}\tau_{1(1,0)}^{(0)}(1,0) - \frac{(4-d)^2}{2(2+c)}\tau_{2(0,1)}^{(0)}(1,0) \right. \\
&\quad \left. + \frac{(4-d)^2}{(d+c)(2+c)(4+c)}\left(3(2+c) + \frac{(4-d)^2}{4}(10+c)\right)\tau_{1(1,0)}^{(0)}(1,0)\right]\theta.
\end{aligned} \tag{33}$$

In (32), (33), $\tau_{1(i,j)}^{(0)}(1,0) = [\partial^{i+j}\tau_1^{(0)}(x,y)/\partial x^i\,\partial y^j]_{x=1,y=0}$ and $\tau_{2(0,1)}^{(0)}(1,0) = [\partial\tau_2^{(0)}(x,y)/\partial y]_{x=1,y=0}$. From (32) we see that on the critical surface $\tau_1^{(0)}(1,0) = t_1^{(0)}(1,0) - t_1^*(1,0) \equiv t^{(0)}(1) = 0$ we have that $\{t_i(l, \psi^2, \theta) \to_{l\to\infty} \{t_i^*(\psi^2, \theta)\}$ only for $d<4$. This implies that the non-Gaussian fixed point is stable only for $2<d<4$ and the corresponding critical exponents are of the spherical model type with $\eta = 0$, $z = 2+c$, $\nu = 1/(d-2)$, $\lambda_1 = (d-2)$, $\lambda_2 = (d-4)$, $\lambda_3 = (d-6-c), \ldots$. Thus, in the present differential formulation, all the ST results (together with the static ones) can be simply reproduced in a more natural way and new explicit results can also be obtained. Studies concerning the global flux in the local parameter space and crossover phenomena require a careful analysis of the RG equations (12) beyond the linear region. However, a direct investigation of the global RG solution is a difficult topic even if, at least in principle, it can be realised by the use of the well known techniques developed for systems of quasi-linear first-order partial equations. Nevertheless, as we shall see in the next section, due to the peculiar structure of the DRG equations, it is possible to obtain additional insights on the global nature of the RG in the large-n limit with an alternative and probably more convenient procedure.

4. Mechanical analogy: a proposal

By inspection of equations (12) and also of the fixed point equations (16) a very peculiar structure appears: the coefficients of the corresponding derivatives are identical. Thus, they constitute a system of equations with the 'same principal part' whose properties are well established (Courant and Hilbert 1962). This surprising characteristic of the DRG transformation in the large n-limit, which emerges clearly only in the differential formulation, gives the possibility to introduce a new interesting way to approach the dynamic (and the static) problem.

We firstly observe that, from a known theory (Courant and Hilbert 1962), valid only for such a special category of systems of first-order partial equations, the system (12) is equivalent to the single homogeneous linear partial equation

$$\frac{\partial \mathcal{S}}{\partial l}+(d-2)(\psi^2-F)\frac{\partial \mathcal{S}}{\partial \psi^2}+(d+c)(\theta-G)\frac{\partial \mathcal{S}}{\partial \theta}+\sum_{i=1}^{2} a_i t_i \frac{\partial \mathcal{S}}{\partial t_i}=0 \tag{34}$$

for a function $\mathcal{S}(l, \psi^2, \theta; t_1, t_2)$, which does not appear explicitly, of the five independent variables l, ψ^2, θ, t_1, t_2. When two 'independent' solutions $\mathcal{S}_i$ $(i=1,2)$ of (34) are known, a solution $\{t_i; i=1,2\}$ of the system (12) can be obtained by solving the algebraic system $\mathcal{S}_i=c_i$ where c_i $(i=1,2)$ are two arbitrary constants. Of course, the fixed point system of equations (16) is equivalent to the single partial equation which is obtained from (34) by using the condition $\partial\mathcal{S}^*/\partial l=0$, where $\mathcal{S}^*=\mathcal{S}^*(\psi^2, \theta, t_1^*, t_2^*)$, with $\mathcal{S}^*(1,0,0,0)=0$, assume the role of a fixed solution of the new DRG equation (34).

Then if we put for convenience $q_1=\psi^2$, $q_2=t_1$, $q_3=\theta$, $q_4=t_2$, $p_j=\partial\mathcal{S}/\partial q_j$ $(j=1,\ldots,4)$ and introduce the function, not depending explicitly on the parameter l,

$$\mathcal{H}(\{q_j\};\{p_j\})=\sum_{j=1}^{4}\alpha_j q_j p_j-[\alpha_1 F(q_2,q_4)p_1+\alpha_3 G(q_2,q_4)p_3] \tag{35}$$

with

$$\alpha_1=d-2, \qquad \alpha_2=2, \qquad \alpha_3=d+c, \qquad \alpha_4=4+c, \tag{36}$$

the equation (34) for $\mathcal{S}(l,\{q_j\})$ can be rewritten as

$$\partial\mathcal{S}/\partial l+\mathcal{H}(\{q_j\};\{\partial\mathcal{S}/\partial q_j\})=0. \tag{37}$$

The corresponding equations of characteristics assume the form

$$\begin{aligned} \mathrm{d}q_j/\mathrm{d}l &= \partial\mathcal{H}/\partial p_j \\ \mathrm{d}p_j/\mathrm{d}l &= -\partial\mathcal{H}/\partial q_j \end{aligned} \qquad (j=1,\ldots,4) \tag{38}$$

and the integration of the original system of quasi-linear partial equations (12) is equivalent to integration of the system or ordinary equations (38).

The results (35)–(38) provide the key for developing the mentioned alternative procedure to investigate the properties of the DRG in the large-n limit from a new point of view. It is evident that (37)–(38) have a structure typical of the Hamilton–Jacobi equation and of Hamilton's canonical equations of motion in classical mechanics, respectively. Then, an interesting analogy between classical mechanics and the DRG in the large-n limit clearly emerges. Indeed, if one looks on the RG parameter l as a 'time-like' variable, $\mathcal{S}(l, \{q_j\})$ can be regarded as the 'action' of an 'equivalent mechanical system' whose 'Hamiltonian' $\mathcal{H}$, defined by (35) in terms of the generalised coordinates $\{q_j\}$ and the conjugate momenta $\{p_j\}$, does not depend explicitly on the 'time' l. With this mechanical analogy, the original problem of the flux in the local parameter space, under iteration of the DRG transformation, can be sought as equivalent to the 'time evolution', in the eight-dimensional 'phase space' $\Omega \equiv (\{q_j\}, \{p_j\})$ of the representative points of the mechanical system described by the equations of motion (38). Of course, since it results that $\partial\mathcal{H}/\partial l \equiv \mathrm{d}\mathcal{H}/\mathrm{d}l = 0$, the 'constant of motion' $\mathcal{H}$ is a quantity 'invariant' under iteration of the DRG transformation. Note that, if we put $q_3 = q_4 = 0$, $q_2 = t$ and

$$\mathcal{S}(l, q_1, q_2, 0, 0) \equiv S(l, \{q_i\}),$$
$$\mathcal{H}(q_1, q_2, 0, 0; p_1, p_2, 0, 0) \equiv H(\{q_i\}; \{p_i\}) = \sum_{i=1}^{2} \alpha_i q_i p_i - \frac{\alpha_1 p_1}{1+q_2}, \tag{39}$$

with $p_i = \partial S/\partial q_i$ $(i = 1, 2)$, equations (34)–(38) reduce just to the corresponding ones for the SRG equation (15). Thus, the mechanical analogy is true also for the statics for which the four-dimensional 'phase space' $\Gamma \equiv (\{q_i\}; \{p_i\})$ is an invariant subspace of the dynamical phase space Ω and the 'Hamiltonian' H is a 'constant of motion'.

Note that the 'mechanical Hamiltonians' $\mathcal{H}$ and H have not the typical structure of usual mechanical systems and as 'constant of motion' cannot be immediately interpreted as mechanical energies. Furthermore, their non-analytical character reflects the highly nonlinear nature of the RG transformation in the large-n limit. In any case, as in classical mechanics, we can separate the 'action' $\mathcal{S}(l, \{q_j\})$ (or $S(l, \{q_i\})$ in the static case) in two parts, one involving $\{q_j\}$ only and the other only the 'time' l:

$$\mathcal{S}(l, \{q_j\}) = \mathcal{S}_0(\{q_j\}) - Kl. \tag{40}$$

Thus, equation (37) reduces to

$$\mathcal{H}(\{q_j\}; \{\partial\mathcal{S}_0/\partial q_j\}) = K, \tag{41}$$

no longer involving the 'time', where K is the (arbitrary) constant value of $\mathcal{H}(\{q_j\}; \{p_j\})$. In the present mechanical analogy, the function $\mathcal{S}_0(\{q_j\})$ assumes the role of 'Hamilton's characteristic function' (the 'reduced action') and, therefore, it generates a canonical transformation in which all the new generalised coordinates are 'cyclic' (Goldstein 1972).

Note that the fixed action $\mathcal{S}^*$ corresponds just to the reduced action when the value $K = 0$ is assumed for the constant of motion $\mathcal{H}$. With the new point of view exposed above, one is now in a position to explore the DRG (and SRG) properties in the large-n limit by using the powerful geometrical and mathematical techniques from the Hamilton–Jacobi theory for classical mechanics (Courant and Hilbert 1962, Stanley 1977, Abraham and Mardsen 1978). Particularly suggestive is the possibility to investigate the nature of the flux in the original parameter space by studying the evolution of the waves of action in the configurational space $\{q_i\}$ with steps very similar to those used for developing the analogy between classical mechanics and geometrical optics (Goldstein 1972, Arnold 1978).

5. Conclusions

On the basis of the results obtained in the previous sections for the dynamics and in I and II for the statics, we think that the proposed differential formulation can be convenient for an advantageous and unified investigation of both the static and dynamic RG properties in the large-n limit. The differential RG equations are, in any case, far simpler and amenable to more analytic solution techniques than the corresponding finite recursion ones. In this scheme many difficulties are circumvented and, apart from the advantages indicated in § 1, we have the concrete possibility to obtain also global results in an explicit form by using less artificial procedures. Furthermore, the structure of the RG in the large-n limit and the strict connection between the statics and the dynamics become more transparent. As regards the mechanical analogy, it may be very usefully utilised for developing an intuitive geometric picture about the nature of the global flux in the parameter space which can give new insights on the RG approach. Note that similar geometric descriptions, based on the general theory of partial differential equations, have been already successfully used in other topics of statistical mechanics (Fisher 1977, Fisher and Au-Yang 1979, Stilck and Salinas 1981, Gartenhaus 1981).

Finally, we wish to point out that the main purpose of § 4 is only to present a new point of view for exploring the RG properties in the large-n limit. A detailed investigation of its implications, based on the Hamilton–Jacobi theory, is under study and we hope to present other aspects of the problem in a future work.

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